

CYCLICITY IN WEIGHTED ℓ^p SPACES

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ABSTRACT. We study the cyclicity in weighted $\ell^p(\mathbb{Z})$ spaces. For $p \geq 1$ and $\beta \geq 0$, let $\ell_\beta^p(\mathbb{Z})$ be the space of sequences $u = (u_n)_{n \in \mathbb{Z}}$ such that $(u_n |n|^\beta) \in \ell^p(\mathbb{Z})$. We obtain both necessary conditions and sufficient conditions for u to be cyclic in $\ell_\beta^p(\mathbb{Z})$, in other words, for $\{(u_{n+k})_{n \in \mathbb{Z}}, k \in \mathbb{Z}\}$ to span a dense subspace of $\ell_\beta^p(\mathbb{Z})$. The conditions are given in terms of the Hausdorff dimension and the capacity of the zero set of the Fourier transform of u .

1. INTRODUCTION AND MAIN RESULTS

For $p \geq 1$ and $\beta \in \mathbb{R}$, we define the Banach space

$$\ell_\beta^p(\mathbb{Z}) = \left\{ u = (u_n)_{n \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}, \quad \|u\|_{\ell_\beta^p}^p = \sum_{n \in \mathbb{Z}} |u_n|^p (1 + |n|)^{p\beta} < \infty \right\}$$

endowed with the norm $\|\cdot\|_{\ell_\beta^p}$. Notice that $\ell_0^p(\mathbb{Z})$ is the classical $\ell^p(\mathbb{Z})$ space.

In this work, we are going to investigate cyclic vectors for $\ell_\beta^p(\mathbb{Z})$ when $\beta \geq 0$. A vector $u \in \ell_\beta^p(\mathbb{Z})$ is called *cyclic* in $\ell_\beta^p(\mathbb{Z})$ if the linear span of $\{(u_{n+k})_{n \in \mathbb{Z}}, k \in \mathbb{Z}\}$ is dense in $\ell_\beta^p(\mathbb{Z})$.

We denote by \mathbb{T} the circle $\mathbb{R}/2\pi\mathbb{Z}$. The Fourier transform of $u \in \ell^p(\mathbb{Z})$ is given by

$$\hat{u} : t \in \mathbb{T} \mapsto \sum_{n \in \mathbb{Z}} u_n e^{int}$$

and when \hat{u} is continuous, we denote by $\mathcal{Z}(\hat{u})$ the zero set on \mathbb{T} of \hat{u} :

$$\mathcal{Z}(\hat{u}) = \{t \in \mathbb{T}, \hat{u}(t) = 0\}.$$

The case $\beta = 0$ was already studied by Wiener, Beurling, Salem and Newman. When $p = 1$ or $p = 2$, Wiener characterized the cyclic vectors u in $\ell^p(\mathbb{Z})$ by the zeros of \hat{u} , with the following theorem.

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Theorem 1.1 ([16]). *Let $u \in \ell^p(\mathbb{Z})$.*

- (1) *If $p = 1$ then u is cyclic in $\ell^1(\mathbb{Z})$ if and only if \hat{u} has no zeros on \mathbb{T} .*
- (2) *If $p = 2$ then u is cyclic in $\ell^2(\mathbb{Z})$ if and only if \hat{u} is non-zero almost everywhere.*

Lev and Olevskii showed that, for $1 < p < 2$ the problem of cyclicity in $\ell^p(\mathbb{Z})$ is more complicated even for sequences in $\ell^1(\mathbb{Z})$. The following Theorem of Lev and Olevskii contradicts the Wiener conjecture.

Theorem 1.2 ([8]). *If $1 < p < 2$, there exist u and v in $\ell^1(\mathbb{Z})$ such that $\mathcal{Z}(\hat{u}) = \mathcal{Z}(\hat{v})$, u is not cyclic in $\ell^p(\mathbb{Z})$, and v is cyclic in $\ell^p(\mathbb{Z})$.*

So we can't characterize the cyclicity of u in $\ell^p(\mathbb{Z})$ in terms of only $\mathcal{Z}(\hat{u})$, the zero set of \hat{u} . However for $u \in \ell^1(\mathbb{Z})$, Beurling, Salem and Newman gave both necessary conditions and sufficient conditions for u to be cyclic in $\ell^p(\mathbb{Z})$. These conditions rely on the "size" of the set $\mathcal{Z}(\hat{u})$ in term of it's h -measure, capacity and Hausdorff dimension.

Given $E \subset \mathbb{T}$ and h a continuous function, non-decreasing and such that $h(0) = 0$, we define the h -measure of E by

$$H_h(E) = \liminf_{\delta \rightarrow 0} \left\{ \sum_{i=0}^{\infty} h(|U_i|), E \subset \bigcup_{i=0}^{\infty} U_i, |U_i| \leq \delta \right\}$$

where the U_i are open intervals of \mathbb{T} and where $|U_i|$ denotes the length of U_i .

The Hausdorff dimension of a subset $E \subset \mathbb{T}$ is given by

$$\dim(E) = \inf\{\alpha \in (0, 1), H_\alpha(E) = 0\} = \sup\{\alpha \in (0, 1), H_\alpha(E) = \infty\},$$

where $H_\alpha = H_h$ for $h(t) = t^\alpha$ (see [6], pp. 23-30).

Let μ be a positive measure on \mathbb{T} and $\alpha \in [0, 1)$. We define the α -energy of μ by

$$I_\alpha(\mu) = \sum_{n \geq 1} \frac{|\hat{\mu}(n)|^2}{(1 + |n|)^{1-\alpha}}.$$

The α -capacity of a Borel set E is given by

$$C_\alpha(E) = 1/\inf\{I_\alpha(\mu), \mu \in \mathcal{M}_{\mathcal{P}}(E)\},$$

where $\mathcal{M}_{\mathcal{P}}(E)$ is the set of all probability measures on \mathbb{T} which are supported on a compact subset of E . If $\alpha = 0$, C_0 is called the logarithmic capacity.

An important property which connects capacity and Hausdorff dimension is that (see [6], p. 34)

$$\dim(E) = \inf\{\alpha \in (0, 1), C_\alpha(E) = 0\} = \sup\{\alpha \in (0, 1), C_\alpha(E) > 0\}. \quad (1.1)$$

In the following theorem, we summarize the results of Beurling [2], Salem [15] (see also [6] pp. 106-110) and Newman [10]. The Hölder conjugate of $p \neq 1$ is noted by $q = \frac{p}{p-1}$.

Theorem 1.3 ([2, 10, 15]). *Let $1 \leq p \leq 2$.*

- (1) *If $u \in \ell^1(\mathbb{Z})$ and $\dim(\mathcal{Z}(\hat{u})) < 2/q$ then u is cyclic in $\ell^p(\mathbb{Z})$.*
- (2) *For $2/q < \alpha \leq 1$, there exists $E \subset \mathbb{T}$ such that $\dim(E) = \alpha$ and every $u \in \ell^1(\mathbb{Z})$ satisfying $\mathcal{Z}(\hat{u}) = E$ is not cyclic in $\ell^p(\mathbb{Z})$.*
- (3) *There exists $E \subset \mathbb{T}$ such that $\dim(E) = 1$ and every $u \in \ell^1(\mathbb{Z})$ satisfying $\mathcal{Z}(\hat{u}) = E$ is cyclic in $\ell^p(\mathbb{Z})$ for all $p > 1$.*

In this paper we give a generalization of the results of Beurling, Salem and Newman to $\ell_\beta^p(\mathbb{Z})$ spaces.

When $\beta q > 1$, we have an analogue of (1) in Wiener's Theorem 1.1: a vector $u \in \ell_\beta^p(\mathbb{Z})$ is cyclic if and only if \hat{u} has no zeros on \mathbb{T} . Indeed, $\ell_\beta^p(\mathbb{Z})$ is a Banach algebra if and only if $\beta q > 1$ (see [4]).

When $p = 2$, Richter, Ross and Sundberg gave a complete characterization of the cyclic vectors u in the weighted harmonic Dirichlet spaces $\ell_\beta^2(\mathbb{Z})$ by showing the following result:

Theorem 1.4 ([14]). *Let $0 < \beta \leq \frac{1}{2}$ and $u \in \ell_\beta^1(\mathbb{Z})$.*

The vector u is cyclic in $\ell_\beta^2(\mathbb{Z})$ if and only if $C_{1-2\beta}(\mathcal{Z}(\hat{u})) = 0$.

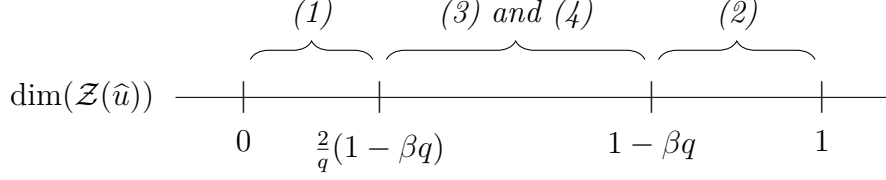
Our first main result is the following theorem.

Theorem A. *Let $1 < p < 2$, $\beta > 0$ such that $\beta q \leq 1$.*

- (1) *If $u \in \ell_\beta^1(\mathbb{Z})$ and $\dim(\mathcal{Z}(\hat{u})) < \frac{2}{q}(1 - \beta q)$ then u is cyclic in $\ell_\beta^p(\mathbb{Z})$.*
- (2) *If $u \in \ell_\beta^1(\mathbb{Z})$ and $\dim(\mathcal{Z}(\hat{u})) > 1 - \beta q$ then u is not cyclic in $\ell_\beta^p(\mathbb{Z})$.*
- (3) *For $\frac{2}{q}(1 - \beta q) < \alpha \leq 1$, there exists a closed subset $E \subset \mathbb{T}$ such that $\dim(E) = \alpha$ and every $u \in \ell_\beta^1(\mathbb{Z})$ satisfying $\mathcal{Z}(\hat{u}) = E$ is not cyclic in $\ell_\beta^p(\mathbb{Z})$.*
- (4) *If $p = \frac{2k}{2k-1}$ for some $k \in \mathbb{N}^*$ there exists a closed subset $E \subset \mathbb{T}$ such that $\dim(E) = 1 - \beta q$ and every $u \in \ell_\beta^1(\mathbb{Z})$ satisfying $\mathcal{Z}(\hat{u}) = E$ is cyclic in $\ell_\beta^p(\mathbb{Z})$.*

Note that in order to prove (2) and (4) we show a stronger result (see Theorem 3.4).

We can summarize Theorem A by the following diagram:



The fourth propriety shows that the bound $1 - \beta q$ obtained in (2) is optimal in the sense that there is no cyclic vector such that $\dim(\mathcal{Z}(\hat{u})) > 1 - q\beta$, and, we can find some cyclic vector u with $\dim(\mathcal{Z}(\hat{u})) = 1 - \beta q$. However this is only proved if $p = \frac{2k}{2k-1}$ for some positive integer k . When p is not of this form, for all positive integer k , we still prove similar results but we loose the optimality because we fail to reach the bound $1 - \beta q$.

The "equality case" $\dim(\mathcal{Z}(\hat{u})) = \frac{2}{q}(1 - \beta q)$ is not treated by the previous theorem. Newman gave a partial answer to this question when $\beta = 0$, by showing that, under some additional conditions on $\mathcal{Z}(\hat{u})$, $\dim(\mathcal{Z}(\hat{u})) = \frac{2}{q}$ implies that u is a cyclic vector (see [10, Theorem 1]). We need the notion of strong α -measure, $\alpha \in (0, 1)$, to state Newman's Theorem in the equality case. For E a compact subset of \mathbb{T} , we note (a_k, b_k) , $k \in \mathbb{N}$ its complementary intervals arranged in non-increasing order of lengths and set

$$r_n = 2\pi - \sum_{k=0}^n (b_k - a_k). \quad (1.2)$$

We will say that E has strong α -measure 0 if

$$\lim_{n \rightarrow \infty} r_n n^{\frac{1}{\alpha}-1} = 0.$$

Notice that if E has strong α -measure 0 then $H_\alpha(E) = 0$. The converse is true for some particular sets like Cantor sets but in general the converse is false (for some countable sets).

Theorem 1.5. *Let $1 < p < 2$ and $u \in \ell^1(\mathbb{Z})$.*

If $\mathcal{Z}(\hat{u})$ has strong α -measure 0 where $\alpha = \frac{2}{q}$ then u is cyclic in $\ell^p(\mathbb{Z})$.

Moreover, in [10], Newman asked the question :

For $u \in \ell^1(\mathbb{Z})$, does $H_{2/q}(\mathcal{Z}(\hat{u})) = 0$ imply that u is cyclic in $\ell^p(\mathbb{Z})$?

A positive answer to this question would contain Theorem 1.1 and Theorem 1.3.(1). We are not able to answer this question completely. Nevertheless, we show that if we replace $2/q$ -measure by h -measure where $h(t) = t^{2/q} \ln(1/t)^{-\gamma}$ with $\gamma > \frac{2}{q}$ then the answer is negative. Moreover we extend Newman's Theorem to $\ell_\beta^p(\mathbb{Z})$.

Theorem B. *Let $1 < p < 2$, $\beta \geq 0$ such that $\beta q < 1$.*

- (1) *If $u \in \ell_\beta^1(\mathbb{Z})$ and $\mathcal{Z}(\hat{u})$ has strong α -measure 0 where $\alpha = \frac{2}{q}(1 - \beta q)$ then u is cyclic in $\ell_\beta^p(\mathbb{Z})$.*
- (2) *For every $\gamma > \frac{2}{q}$, there exists a closed subset $E \subset \mathbb{T}$ such that every $u \in \ell_\beta^1(\mathbb{Z})$ satisfying $\mathcal{Z}(\hat{u}) = E$ is not cyclic in $\ell_\beta^p(\mathbb{Z})$ and such that $H_h(E) = 0$ where $h(t) = t^\alpha \ln(e/t)^{-\gamma}$ with $\alpha = \frac{2}{q}(1 - \beta q)$*

Note that the set E constructed in part (2) of Theorem B satisfy $\dim(E) = \frac{2}{q}(1 - \beta q)$.

2. PRELIMINARIES AND LEMMAS

Let $1 \leq p < \infty$ and $\beta \in \mathbb{R}$. We denote by $\mathcal{D}'(\mathbb{T})$ the set of distributions on \mathbb{T} and $\mathcal{M}(\mathbb{T})$ the set of measures on \mathbb{T} . For $S \in \mathcal{D}'(\mathbb{T})$, we denote by $\hat{S} = (\hat{S}(n))_{n \in \mathbb{Z}}$ the sequence of Fourier coefficients of S and we write $S = \sum_n \hat{S}(n) e_n$, where $e_n(t) = e^{int}$. The space $A_\beta^p(\mathbb{T})$ will be the set of all distributions $S \in \mathcal{D}'(\mathbb{T})$ such that \hat{S} belongs to $\ell_\beta^p(\mathbb{Z})$. We endow $A_\beta^p(\mathbb{T})$ with the norm $\|S\|_{A_\beta^p(\mathbb{T})} = \|\hat{S}\|_{\ell_\beta^p}$. We will write $A^p(\mathbb{T})$ for the space $A_0^p(\mathbb{T})$. Thus the Fourier transformation is an isometric isomorphism between $\ell_\beta^p(\mathbb{Z})$ and $A_\beta^p(\mathbb{T})$. We prefer to work with $A_\beta^p(\mathbb{T})$ rather than $\ell_\beta^p(\mathbb{Z})$. In this section we establish some properties of $A_\beta^p(\mathbb{T})$ which will be needed to prove Theorems A and B.

For $1 \leq p < \infty$ and $\beta \geq 0$ we define the product of $f \in A_\beta^1(\mathbb{T})$ and $S \in A_\beta^p(\mathbb{T})$ by

$$fS = \sum_{n \in \mathbb{Z}} (\hat{f} * \hat{S})(n) e_n = \sum_{n \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} \hat{f}(k) \hat{S}(n - k) \right) e_n,$$

and we see that $\|fS\|_{A_\beta^p(\mathbb{T})} \leq \|f\|_{A_\beta^1(\mathbb{T})} \|S\|_{A_\beta^p(\mathbb{T})}$. Note that if $S \in A_{-\beta}^p(\mathbb{T})$ we can also define the product $fS \in A_{-\beta}^p(\mathbb{T})$ by the same formula and obtain a similar inequality: $\|fS\|_{A_{-\beta}^p(\mathbb{T})} \leq \|f\|_{A_\beta^1(\mathbb{T})} \|S\|_{A_{-\beta}^p(\mathbb{T})}$.

For $p \neq 1$, the dual space of $A_\beta^p(\mathbb{T})$ can be identified with $A_{-\beta}^q(\mathbb{T})$ ($q = \frac{p}{p-1}$) by the following formula

$$\langle S, T \rangle = \sum_{n \in \mathbb{Z}} \hat{S}(n) \hat{T}(-n), \quad S \in A_\beta^p(\mathbb{T}), \quad T \in A_{-\beta}^q(\mathbb{T}).$$

We denote by $\mathcal{P}(\mathbb{T})$ the set of trigonometric polynomials on \mathbb{T} . We rewrite the definition of cyclicity in the spaces $A_\beta^p(\mathbb{T})$ for $\beta \geq 0$: $S \in$

$A_\beta^p(\mathbb{T})$ will be a cyclic vector if the set $\{PS, P \in \mathcal{P}(\mathbb{T})\}$ is dense in $A_\beta^p(\mathbb{T})$. It's clear that the cyclicity of S in $A_\beta^p(\mathbb{T})$ is equivalent to the cyclicity of the sequence \hat{S} in $\ell_\beta^p(\mathbb{Z})$. Moreover for $1 \leq p < \infty$ and $\beta \geq 0$, S is cyclic in $A_\beta^p(\mathbb{T})$ if and only if there exists a sequence (P_n) of trigonometric polynomials such that

$$\lim_{n \rightarrow \infty} \|1 - P_n S\|_{A_\beta^p(\mathbb{T})} = 0. \quad (2.1)$$

We need the following lemmas which gives us different inclusions between the $A_\beta^p(\mathbb{T})$ spaces.

Lemma 2.1. *Let $1 \leq r, s < \infty$ and $\beta, \gamma \in \mathbb{R}$.*

- (1) *If $r \leq s$ then $A_\beta^r(\mathbb{T}) \subset A_\gamma^s(\mathbb{T}) \Leftrightarrow \gamma \leq \beta$.*
- (2) *If $r > s$ then $A_\beta^r(\mathbb{T}) \subset A_\gamma^s(\mathbb{T}) \Leftrightarrow \beta - \gamma > \frac{1}{s} - \frac{1}{r}$.*

Proof. (1) : We suppose that $r \leq s$. If $\gamma \leq \beta$ and $S \in A_\beta^r(\mathbb{T})$, we have

$$\sum_{n \in \mathbb{Z}} |\hat{S}(n)|^s (1 + |n|)^{\gamma s} \leq \sum_{n \in \mathbb{Z}} |\hat{S}(n)|^s (1 + |n|)^{\beta s}.$$

Since $\|\cdot\|_{\ell^s} \leq \|\cdot\|_{\ell^r}$, we obtain $S \in A_\gamma^s(\mathbb{T})$ and so $A_\beta^r(\mathbb{T}) \subset A_\gamma^s(\mathbb{T})$.

Now suppose $\gamma > \beta$. Let $S \in \mathcal{D}'(\mathbb{T})$ be given by

$$\hat{S}(n)(1 + |n|)^\beta = \begin{cases} (1 + m)^{-2/r} & \text{if } |n| = 2^m \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $S \in A_\beta^r(\mathbb{T}) \setminus A_\gamma^s(\mathbb{T})$.

(2) : Now suppose that $r > s$. If $\beta - \gamma > \frac{1}{s} - \frac{1}{r}$, we have by Hölder's inequality,

$$\|S\|_{A_\gamma^s(\mathbb{T})} \leq \|S\|_{A_\beta^r(\mathbb{T})} \left(\sum_{n \in \mathbb{Z}} (1 + |n|)^{\frac{rs}{r-s}(\gamma-\beta)} \right)^{1-s/r}, \quad S \in A_\beta^r(\mathbb{T}),$$

so that $A_\beta^r(\mathbb{T}) \subset A_\gamma^s(\mathbb{T})$.

Now suppose that $\beta - \gamma < \frac{1}{s} - \frac{1}{r}$. Let $\varepsilon > 0$ such that $\beta - \gamma + \varepsilon < \frac{1}{s} - \frac{1}{r}$, $\alpha = -\frac{1}{s} - \gamma + \varepsilon$ and let $S \in \mathcal{D}'(\mathbb{T})$ be such that $\hat{S}(n) = n^\alpha$. We have $S \in A_\beta^r(\mathbb{T}) \setminus A_\gamma^s(\mathbb{T})$.

For the case $\beta - \gamma = \frac{1}{s} - \frac{1}{r}$ we take $S \in \mathcal{D}'(\mathbb{T})$ such that

$$\hat{S}(n)^r (1 + |n|)^{\beta r} = \frac{1}{(1 + |n|) \ln(1 + |n|)^{1+\varepsilon}}$$

with $\varepsilon = \frac{r}{s} - 1 > 0$. We can show that $S \in A_\beta^r(\mathbb{T}) \setminus A_\gamma^s(\mathbb{T})$ which proves that $A_\beta^r(\mathbb{T}) \not\subset A_\gamma^s(\mathbb{T})$. \square

For $E \subset \mathbb{T}$, we denote by $A_\beta^p(E)$ the set of $S \in A_\beta^p(\mathbb{T})$ such that $\text{supp}(S) \subset E$, where $\text{supp}(S)$ denotes the support of the distribution S . The following lemma is a direct consequence of the definition of capacity (see [6]) and the inclusion $A_{-\beta}^q(\mathbb{T}) \subset A_{\frac{\alpha-1}{2}}^2(\mathbb{T})$ when $q \geq 2$ and $0 \leq \alpha < \frac{2}{q}(1 - \beta q)$.

Lemma 2.2. *Let E a Borel set, $\beta \geq 0$ and $q \geq 2$. If there exists α , $0 \leq \alpha < \frac{2}{q}(1 - \beta q)$, such that $C_\alpha(E) = 0$ then $A_{-\beta}^q(E) = \{0\}$.*

We obtain the first results about cyclicity for the spaces $A_\beta^p(\mathbb{T})$, when $A_\beta^p(\mathbb{T})$ is a Banach algebra. More precisely, we have (see [4])

Proposition 2.3. *Let $1 \leq p < \infty$ and $\beta \geq 0$. $A_\beta^p(\mathbb{T})$ is a Banach algebra if and only if $\beta q > 1$. Moreover when $\beta q > 1$, a vector $f \in A_\beta^p(\mathbb{T})$ is cyclic in $A_\beta^p(\mathbb{T})$ if and only if f has no zeros on \mathbb{T} .*

Let $f \in A_\beta^1(\mathbb{T})$ and $S \in \mathcal{D}'(\mathbb{T})$. We denote by $\mathcal{Z}(f)$ the zero set of the function f . Recall that $e_n : t \mapsto e^{int}$.

Lemma 2.4. *Let $1 \leq p < \infty$ and $\beta \geq 0$. Let $f \in A_\beta^1(\mathbb{T})$ and $S \in A_{-\beta}^p(\mathbb{T})$. If for all $n \in \mathbb{Z}$, $\langle S, e_n f \rangle = 0$ then $\text{supp}(S) \subset \mathcal{Z}(f)$.*

Proof. We have

$$\langle S, e_n f \rangle = \langle f S, e_n \rangle = 0.$$

Hence $f S = 0$. Let $\varphi \in C^\infty(\mathbb{T})$ such that $\text{supp}(\varphi) \subset \mathbb{T} \setminus \mathcal{Z}(f)$. We claim that $\frac{\varphi}{f} \in A_\beta^1(\mathbb{T}) \subset A_\beta^q(\mathbb{T})$ where $q = \frac{p}{p-1}$. So we obtain

$$\langle S, \varphi \rangle = \langle f S, \frac{\varphi}{f} \rangle = 0$$

which proves that $\text{supp}(S) \subset \mathcal{Z}(f)$.

Now we prove the claim. Let $\varepsilon = \min\{|f(t)|, t \in \text{supp}(\varphi)\} > 0$ and $P \in \mathcal{P}(\mathbb{T})$ such that $\|f - P\|_{A_\beta^1(\mathbb{T})} \leq \varepsilon/3$.

By the Cauchy-Schwarz and Parseval inequalities, for every $g \in C^1(\mathbb{T})$, we get

$$\|g\|_{A_\beta^1(\mathbb{T})} \leq \|g\|_\infty + 2\sqrt{\frac{2-2\beta}{1-2\beta}}\|g'\|_\infty. \quad (2.2)$$

Now, as in [11], by applying (2.2) to $\frac{\varphi}{P^n}$ we see that

$$\frac{\varphi}{f} = \sum_{n \geq 1} \varphi \frac{(P - f)^{n-1}}{P^n} \in A_\beta^1(\mathbb{T}),$$

which finishes the proof. \square

Proposition 2.5. *Let $1 \leq p < \infty$ and $f \in A_\beta^1(\mathbb{T})$ with $\beta \geq 0$. We have*

- (1) *If f is not cyclic in $A_\beta^p(\mathbb{T})$ then there exists $S \in A_{-\beta}^q(\mathbb{T}) \setminus \{0\}$ such that $\text{supp}(S) \subset \mathcal{Z}(f)$.*
- (2) *If there exists a nonzero measure $\mu \in A_{-\beta}^q(\mathbb{T})$ such that $\text{supp}(\mu) \subset \mathcal{Z}(f)$ then f is not cyclic in $A_\beta^p(\mathbb{T})$.*

Proof. (1) If f is not cyclic in $A_\beta^p(\mathbb{T})$, by duality there exists $S \in A_{-\beta}^q(\mathbb{T}) \setminus \{0\}$ such that

$$\langle S, e_n f \rangle = 0, \quad \forall n \in \mathbb{Z}.$$

Thus, by lemma 2.4, we have $\text{supp}(S) \subset \mathcal{Z}(f)$.

(2) Let $\mu \in A^q(\mathbb{T}) \cap \mathcal{M}(\mathbb{T}) \setminus \{0\}$ such that $\text{supp}(\mu) \subset \mathcal{Z}(f)$. Since μ is a measure on \mathbb{T} we have $\langle \mu, e_n f \rangle = 0$, for all $n \in \mathbb{Z}$. So f is not cyclic in $A_\beta^p(\mathbb{T})$. \square

Recall that $A_\beta^1(\mathbb{T})$ is a Banach algebra. Let I be a closed ideal in $A_\beta^1(\mathbb{T})$. We denote by \mathcal{Z}_I the set of common zeros of the functions of I ,

$$\mathcal{Z}_I = \bigcap_{f \in I} \mathcal{Z}(f).$$

We have the following result about spectral synthesis in $A_\beta^1(\mathbb{T})$.

Lemma 2.6. *Let $0 \leq \beta < 1/2$. Let I be a closed ideal in $A_\beta^1(\mathbb{T})$. If g is a Lipschitz function which vanishes on \mathcal{Z}_I then $g \in I$.*

Proof. The proof is similar to the one given in [6] pp. 121-123. For the sake of completeness we give the important steps. Let I^\perp be the set of all S in the dual space of $A_\beta^1(\mathbb{T})$ satisfying $\langle S, f \rangle = 0$ for all $f \in I$. Let g be a Lipschitz function which vanishes on \mathcal{Z}_I and $S \in I^\perp$. By Lemma 2.4, $\text{supp}(S) \subset \mathcal{Z}_I$. For $h > 0$, we set $S_h = \widehat{S} * \Delta_h$ where $\Delta_h : t \mapsto \frac{-|t|}{h^2} + \frac{1}{h}$ if $t \in [-h, h]$ and 0 otherwise. We have $\widehat{\Delta_h}(0) = 1/2\pi$ and $\widehat{\Delta_h}(n) = \frac{1}{2\pi} \frac{4 \sin(nh/2)^2}{(nh)^2}$ for $n \neq 0$. Since S is in the dual of $A_\beta^1(\mathbb{T})$, $S_h \in A^1(\mathbb{T})$. Moreover we have $\text{supp}(S_h) \subset \text{supp}(S) + \text{supp}(\Delta_h) \subset \mathcal{Z}_I^h := \mathcal{Z}_I + [-h, h]$. We have

$$\begin{aligned} |\langle S_h, g \rangle|^2 &= \left| \int_{\mathcal{Z}_I^h \setminus \mathcal{Z}(g)} S_h(x) g(x) dx \right|^2 \\ &\leq \left(\sum_{n \in \mathbb{Z}} |\widehat{S}(n) \widehat{\Delta_h}(n)|^2 \right) \left(\int_{\mathcal{Z}_I^h \setminus \mathcal{Z}(g)} |g(x)|^2 dx \right) \\ &\leq C \left(\sum_{n \in \mathbb{Z}} \frac{\widehat{S}(n)^2}{n^2} \right) (|\mathcal{Z}_I^h \setminus \mathcal{Z}(g)|) \end{aligned}$$

where C is a positive constant and where $|E|$ denotes the Lebesgue measure of E . So $\lim_{h \rightarrow 0} \langle S_h, g \rangle = 0$. By the dominated convergence theorem, we obtain that

$$\lim_{h \rightarrow 0} \langle S_h, g \rangle = \lim_{h \rightarrow 0} \sum_{n \in \mathbb{Z}} \widehat{S}_h(n) \widehat{g}(-n) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \widehat{S}(n) \widehat{g}(-n) = \frac{1}{2\pi} \langle S, g \rangle.$$

So $\langle S, g \rangle = 0$. Therefore $g \in I$. \square

We also need the following lemma which is a consequence of Lemma 2.6. Newman gave a proof of this when $\beta = 0$ (see [10, Lemma 2]).

Lemma 2.7. *Let $0 \leq \beta < 1/2$ and a closed set $E \subset \mathbb{T}$. There exists (f_n) a sequence of Lipschitz functions which are zero on E and such that*

$$\lim_{n \rightarrow \infty} \|f_n - 1\|_{A_\beta^p(\mathbb{T})} = 0$$

if and only if every $f \in A_\beta^1(\mathbb{T})$ satisfying $\mathcal{Z}(f) = E$ is cyclic in $A_\beta^p(\mathbb{T})$.

3. PROOF OF THEOREM A

Before proving Theorem A, let us recall Salem's Theorem (see [15] and [6] pp. 106-110).

Theorem 3.1. *Let $0 < \alpha < 1$ and $q > \frac{2}{\alpha}$. There exists a compact set $E \subset \mathbb{T}$ which satisfies $\dim(E) = \alpha$ and there exists a positive measure $\mu \in A^q(\mathbb{T}) \setminus \{0\}$ such that $\text{supp}(\mu) \subset E$.*

To prove Theorem A, we also need the following result. The case $\beta = 0$ was considered by Newman in [10]. For $k \in \mathbb{N}$ and $E \subset \mathbb{T}$, we denote

$$k \times E = E + E + \dots + E = \left\{ \sum_{n=1}^k x_n, x_n \in E \right\}.$$

Theorem 3.2. *Let $1 < p < 2$ and $\beta > 0$ such that $\beta q \leq 1$, and let $f \in A_\beta^1(\mathbb{T})$.*

- (a) *Let $k \in \mathbb{N}^*$ be such that $k \leq q/2$. If $C_\alpha(k \times \mathcal{Z}(f)) = 0$ for some $\alpha < \frac{2}{q}(1 - \beta q)k$, then f is cyclic in $A_\beta^p(\mathbb{T})$.*
- (b) *Let $k \in \mathbb{N}^*$ be such that $q/2 \leq k \leq 1/(2\beta)$. If $C_\alpha(k \times \mathcal{Z}(f)) = 0$ where $\alpha = 1 - 2k\beta$, then f is cyclic in $A_\beta^p(\mathbb{T})$.*

Proof. Let $k \in \mathbb{N}^*$. Suppose that f is not cyclic in $A_\beta^p(\mathbb{T})$. Then there exists $L \in A_{-\beta}^q(\mathbb{T})$, the dual of $A_\beta^p(\mathbb{T})$, such that $L(1) = 1$ and $L(Pf) = 0$, for all $P \in \mathcal{P}(\mathbb{T})$.

Since $\beta < \frac{1}{2}$, by (2.2), we get $C^1(\mathbb{T}) \subset A_\beta^1(\mathbb{T}) \subset A_\beta^p(\mathbb{T})$, and by [9] (see also [10, Lemma 5]), there exists $\phi \in L^2(\mathbb{T})$ such that

$$L(g) = \int_{\mathbb{T}} (g'(x)\phi(x) + g(x)) \, dx, \quad \forall g \in C^1(\mathbb{T}).$$

Since $L \in A_{-\beta}^q(\mathbb{T})$ which implies $(L(e_n))_{n \in \mathbb{Z}} \in \ell_{-\beta}^q(\mathbb{Z})$, we obtain

$$\sum_{n \in \mathbb{Z}} |n\widehat{\phi}(n)|^q (1 + |n|)^{-\beta q} < \infty. \quad (3.1)$$

Moreover we have,

$$\int_{\mathbb{T}} ((e_n f)'(x)\phi(x) + (e_n f)(x)) \, dx = 0, \quad n \in \mathbb{Z},$$

and so $\langle \phi' - 1, e_n f \rangle = 0$ where ϕ' is defined in terms of distribution. By (3.1), $\phi' - 1 \in A_{-\beta}^q(\mathbb{T})$, so by lemma 2.4, we get $\text{supp}(\phi' - 1) \subset \mathcal{Z}(f)$.

For $m \in \mathbb{N}$, we denote by ϕ^{*m} the result of convolving ϕ with itself m times. Using the fact that $S' * T = S * T'$ and $1 * S' = 0$ for any distributions S and T , we have

$$(\phi' - 1) * \left((\phi^{*(m-1)})^{(m-1)} + (-1)^{m-1} \right) = (\phi^{*m})^{(m)} + (-1)^m.$$

So we can show by induction on $m \geq 1$ and by the formula $\text{supp}(T * S) \subset \text{supp}(T) + \text{supp}(S)$ that

$$\text{supp} \left((\phi^{*m})^{(m)} + (-1)^m \right) \subset m \times \mathcal{Z}(f), \quad \forall m \geq 1. \quad (3.2)$$

Note that $\widehat{(\phi^{*k})^{(k)}}(n) = i^k n^k \widehat{\phi}(n)^k$ for $k \geq 1$ and $n \in \mathbb{Z}$.

(a) : Suppose that $0 < k \leq q/2$ and $C_\alpha(k \times \mathcal{Z}(f)) = 0$ for some $\alpha < \frac{2}{q}(1 - \beta q)k$. We rewrite (3.1) as

$$\sum_{n \in \mathbb{Z}} \left(|n\widehat{\phi}(n)|^k \right)^{\frac{q}{k}} (1 + |n|)^{-\frac{q}{k}\beta k} < \infty.$$

So, if we set $q' = \frac{q}{k} \geq 2$ and $\beta' = \beta k$, we have $(\phi^{*k})^{(k)} \in A_{-\beta'}^{q'}(\mathbb{T})$. By (3.2) and by Lemma 2.2 we obtain that $(\phi^{*k})^{(k)} = (-1)^{k-1}$. This contradicts the fact that $\widehat{(\phi^{*k})^{(k)}}(0) = 0$.

(b) : Now suppose that $k \geq q/2$ and $C_\alpha(k \times \mathcal{Z}(f)) = 0$ where $\alpha = 1 - 2k\beta$. Since $q \leq 2k$, we have by (3.1),

$$\sum_{n \in \mathbb{Z}} |n\widehat{\phi}(n)|^{2k} (1 + |n|)^{-2k\beta} < \infty.$$

So $(\phi^{*k})^{(k)} \in A_{-k\beta}^2(\mathbb{T})$ and $(\phi^{*k})^{(k)} = (-1)^{k-1}$. Again this is absurd since $(\widehat{\phi^{*k}})^{(k)}(0) = 0$. \square

We need to compute the capacity of the Minkowski sum of some Cantor type subset of \mathbb{T} . We denote by $[x]$ the integer part of $x \in \mathbb{R}$. For $\lambda \in [0, 1]$ and $k \in \mathbb{N}^*$, we define

$$K_\lambda^k = \{m \in \mathbb{N}, \exists j \in \mathbb{N}, m \in [2^j, 2^j(1 + \lambda + 1/j) - k + 1]\}$$

and we set in $\mathbb{R}/\mathbb{Z} \simeq [0, 1[$,

$$S_\lambda^k = \left\{ x = \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}}, (x_i) \in \{0, 1\}^{\mathbb{N}} \text{ such that } i \in K_\lambda^k \Rightarrow x_i = 0 \right\}.$$

We denote $K_\lambda = K_\lambda^1$ and $S_\lambda = S_\lambda^1$.

To prove (4) of Theorem A we need the following lemma.

Lemma 3.3. *For all $k \geq 1$, we have*

- (1) $k \times S_\lambda \subset S_\lambda^k$;
- (2) $C_\alpha(S_\lambda^k) = 0$ if and only if $\alpha \geq \frac{1-\lambda}{1+\lambda}$;
- (3) $\dim(k \times S_\lambda) = \frac{1-\lambda}{1+\lambda}$ and $C_{\frac{1-\lambda}{1+\lambda}}(k \times S_\lambda) = 0$.

Proof. (1) : We prove this by induction. If $k = 1$ we have $S_\lambda = S_\lambda^1$. We suppose the result true for $k - 1$ for some $k \geq 2$ and we will show $k \times S_\lambda \subset S_\lambda^k$. We have $k \times S_\lambda \subset (k - 1) \times S_\lambda + S_\lambda \subset S_\lambda^{k-1} + S_\lambda$. Let $x \in S_\lambda^{k-1}$, $y \in S_\lambda$ and $z = x + y$. Denote by (x_i) , (y_i) and (z_i) their binary decomposition. Let $m \in K_\lambda^k$. There exists $j \in \mathbb{N}$ such that $m \in [2^j, 2^j(1 + \lambda + 1/j) - k + 1]$. Since $m \in K_\lambda^k$, m and $m + 1$ are contained in $K_\lambda^{k-1} \subset K_\lambda$, we have $x_m = y_m = x_{m+1} = y_{m+1} = 0$. Therefore we write

$$z = x + y = \sum_{i=0}^{m-1} \frac{x_i + y_i}{2^{i+1}} + \sum_{i=m+2}^{\infty} \frac{x_i + y_i}{2^{i+1}}.$$

Note that for infinitely many $i \geq m + 2$, $x_i + y_i < 2$, so we see that

$$\sum_{i=m+2}^{\infty} \frac{x_i + y_i}{2^{i+1}} < \frac{1}{2^{m+1}}.$$

Therefore, we obtain by uniqueness of the decomposition that $z_m = 0$. This proves that $x + y \in S_\lambda^k$ and $k \times S_\lambda \subset S_\lambda^k$.

(2) : We will study the capacity of S_λ^k by decomposing it. First we show that the set S_λ^k is a generalized Cantor set in the sense of [3, 13].

Let $\nu_j = [2^j(1 + \lambda + 1/j) - k + 1] + 1$ and N_0 (depending only on k and λ) such that for all $j \geq N_0$, $2^j < \nu_j < 2^{j+1}$. We set for $N \geq N_0$,

$$l_N = \sum_{j=N}^{\infty} \frac{1}{2^{\nu_j}} - \frac{1}{2^{2^{j+1}}}.$$

Since $2^j(1 + \lambda + 1/j) - k + 1 < \nu_j \leq 2^j(1 + \lambda + 1/j) - k + 2$, we have

$$\begin{aligned} \sum_{j=N}^{\infty} \frac{1}{2^{2^j(1+\lambda+\frac{1}{j})}} \left(\frac{1}{2^{2-k}} - \frac{1}{2^{2^j(1-\lambda-\frac{1}{j})}} \right) &\leq l_N \\ &\leq \sum_{j=N}^{\infty} \frac{1}{2^{2^j(1+\lambda+\frac{1}{j})}} \left(\frac{1}{2^{1-k}} - \frac{1}{2^{2^j(1-\lambda-\frac{1}{j})}} \right) \end{aligned}$$

On one hand, there exists $C \geq 1$ such that for all $j \geq N$,

$$\frac{1}{C} \leq \frac{1}{2^{2-k}} - \frac{1}{2^{2^j(1-\lambda-\frac{1}{j})}} \leq \frac{1}{2^{1-k}} - \frac{1}{2^{2^j(1-\lambda-\frac{1}{j})}} \leq C.$$

On the other hand, for $N \geq N_0$,

$$\begin{aligned} \frac{1}{2^{2^N(1+\lambda+\frac{1}{N})}} &\leq \sum_{j=N}^{\infty} \frac{1}{2^{2^j(1+\lambda+\frac{1}{j})}} \leq \frac{1}{2^{2^N(1+\lambda+\frac{1}{N})}} + \sum_{j=0}^{\infty} \left(\frac{1}{2^{2^{N+1}(1+\lambda)}} \right)^{2^j} \\ &\leq \frac{1}{2^{2^N(1+\lambda+\frac{1}{N})}} + \sum_{j=0}^{\infty} \left(\frac{1}{2^{2^{N+1}(1+\lambda)}} \right)^{j+1} \\ &\leq \frac{1}{2^{2^N(1+\lambda+\frac{1}{N})}} + \frac{2}{2^{2^{N+1}(1+\lambda)}} \\ &\leq \frac{3}{2^{2^N(1+\lambda+\frac{1}{N})}}. \end{aligned}$$

Hence we obtain that l_N is comparable to $2^{-2^N(1+\lambda+1/N)}$, that is:

$$\frac{1}{C 2^{2^N(1+\lambda+\frac{1}{N})}} \leq l_N \leq \frac{3C}{2^{2^N(1+\lambda+\frac{1}{N})}}. \quad (3.3)$$

Moreover we have

$$l_N = \frac{1}{2^{\nu_N}} - \sum_{j=N+1}^{\infty} \frac{1}{2^{2^j}} - \frac{1}{2^{\nu_j}} < \frac{1}{2^{\nu_N}} \leq \frac{1}{2^{2^N}}. \quad (3.4)$$

We set

$$E_N = \left\{ \sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} + l_N z, \ z \in [0, 1], \ x_i \in \{0, 1\}, \ i \in K_{\lambda}^k \Rightarrow x_i = 0 \right\}.$$

We can see E_N as a union of disjoint intervals by writing

$$E_N = \bigcup_{\substack{(x_i) \in \{0,1\}^{2^N} \\ i \in K_\lambda^k \Rightarrow x_i = 0}} E_N^{(x_i)},$$

where

$$E_N^{(x_i)} = \sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} + l_N[0, 1].$$

Note that the intervals $E_N^{(x_i)}$ are disjoint since by (3.4), $l_N < \frac{1}{2^{2^N}}$. For fixed $N \geq N_0$, let $(x_i)_{0 \leq i \leq 2^N-1} \in \{0,1\}^{2^N}$ and $(y_i)_{0 \leq i \leq 2^{N+1}-1} \in \{0,1\}^{2^{N+1}}$.

Claim : $E_{N+1}^{(y_i)} \subset E_N^{(x_i)}$ if and only if $x_i = y_i$ for all $0 \leq i < 2^N$ and $y_i = 0$ for all $2^N \leq i < 2^{N+1}$.

Indeed, suppose that $E_{N+1}^{(y_i)} \subset E_N^{(x_i)}$ and let $u \in E_{N+1}^{(y_i)}$. We have

$$u = \sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} + l_N z_1 = \sum_{i=0}^{2^{N+1}-1} \frac{y_i}{2^{i+1}} + l_{N+1} z_2,$$

where z_1 and z_2 are in $[0, 1]$. By (3.4), $l_N < \frac{1}{2^{\nu_N}}$, and using the uniqueness of the binary representation, we obtain $x_i = y_i$ for all $0 \leq i < 2^N$ and $y_i = 0$ for all $2^N \leq i < 2^{N+1}$.

Now suppose $x_i = y_i$ for all $0 \leq i < 2^N$ and $y_i = 0$ for all $2^N \leq i < 2^{N+1}$. Let $u \in E_{N+1}^{(y_i)}$. We write

$$u = \sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} + \sum_{i=2^N}^{2^{N+1}-1} \frac{y_i}{2^{i+1}} + l_{N+1} z,$$

where $z \in [0, 1]$. Note that

$$\sum_{i=2^N}^{2^{N+1}-1} \frac{1}{2^{i+1}} + l_{N+1} = \frac{1}{2^{\nu_N}} - \frac{1}{2^{2^{N+1}}} + l_{N+1} = l_N. \quad (3.5)$$

So we have

$$\sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} \leq \sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} + \sum_{i=2^N}^{2^{N+1}-1} \frac{y_i}{2^{i+1}} + l_{N+1} z \leq \frac{1}{2} \sum_{i=0}^{2^N-1} \frac{x_i}{2^i} + l_N,$$

and $u \in E_N^{(x_i)}$. This conclude the proof of the claim.

By the claim, for fixed (x_i) and for $N \geq N_0$, we have the following properties :

(i) the interval $E_N^{(x_i)}$ contains precisely

$$p_N = \#\{(y_i)_{\nu_N \leq i \leq 2^{N+1}-1} : y_i \in \{0, 1\}\} = 2^{2^{N+1}-\nu_N}$$

intervals of the form $E_{N+1}^{(y_i)}$,

(ii) the intervals of the form $E_{N+1}^{(y_i)}$ contained in $E_N^{(x_i)}$ are equidistant intervals of length l_{N+1} : the distance of two consecutive intervals of the form $E_{N+1}^{(y_i)}$ is equal to $\frac{1}{2^{2^{N+1}}-l_{N+1}}$,

(iii) if we denote $E_N^{(x_i)} = [a, b]$ then there exist (y_i) and (z_i) such that $E_{N+1}^{(y_i)} = [a, a + l_{N+1}]$ and $E_{N+1}^{(z_i)} = [b - l_{N+1}, b]$.

Finally we can write S_λ^k as

$$S_\lambda^k = \bigcap_{N \geq N_0} E_N.$$

This shows that S_λ^k is a generalized Cantor set in the sense of [3, 13]. So, by [3, 13], we have for $0 < \alpha < 1$ that $C_\alpha(S_\lambda^k) = 0$ if and only if

$$\sum_{N=N_0}^{\infty} \frac{1}{(p_{N_0} \cdots p_{N-1}) l_N^\alpha} = \infty.$$

Since

$$\begin{aligned} 2^{(k-2)(N-N_0)+(2^N-2^{N_0})(1-\lambda)-\sigma_N} &\leq p_{N_0} \cdots p_{N-1} \\ &\leq 2^{(k-1)(N-N_0)+(2^N-2^{N_0})(1-\lambda)-\sigma_N}, \end{aligned}$$

where

$$\sigma_N = \sum_{j=N_0}^{N-1} \frac{2^j}{j},$$

we have, by (3.3), $C_\alpha(S_\lambda^k) = 0$ if and only if

$$\sum_{N=N_0}^{\infty} 2^{2^N(\alpha(1+\lambda)-(1-\lambda))+\alpha 2^N/N+\sigma_N-(k-1)(N-N_0)+2^{N_0}(1-\lambda)} = \infty.$$

Therefore $C_\alpha(S_\lambda^k) = 0$ if and only if $\alpha \geq \frac{1-\lambda}{1+\lambda}$.

(3) immediately follows from (1) and (2) by the capacity property (1.1). \square

We are ready to prove Theorem A. The following Theorem is a reformulation of Theorem A in $A_\beta^p(\mathbb{T})$ spaces.

Theorem 3.4. *Let $1 < p < 2$, $\beta > 0$ such that $\beta q \leq 1$.*

(1) If $f \in A_\beta^1(\mathbb{T})$ and $\dim(\mathcal{Z}(f)) < \frac{2}{q}(1 - \beta q)$ then f is cyclic in $A_\beta^p(\mathbb{T})$.

- (2) If $f \in A_\beta^1(\mathbb{T})$ and $C_{1-\beta q}(\mathcal{Z}(f)) > 0$ then f is not cyclic in $A_\beta^p(\mathbb{T})$.
- (3) For $\frac{2}{q}(1 - \beta q) < \alpha \leq 1$, there exists a closed set $E \subset \mathbb{T}$ such that $\dim(E) = \alpha$ and every $f \in A_\beta^1(\mathbb{T})$ satisfying $\mathcal{Z}(f) = E$ is not cyclic in $A_\beta^p(\mathbb{T})$.
- (4) Let $k = [q/2]$. For all $\varepsilon > 0$, there exists a closed set $E \subset \mathbb{T}$ such that

$$\dim(E) \geq \max \left(\frac{2}{q}(1 - \beta q)k - \varepsilon, 1 - 2(k + 1)\beta \right)$$

and such that every $f \in A_\beta^1(\mathbb{T})$ satisfying $\mathcal{Z}(f) = E$ is cyclic in $A_\beta^p(\mathbb{T})$.

Furthermore, if $p = \frac{2k}{2k-1}$ for some $k \in \mathbb{N}^*$, E can be chosen such that $\dim(E) = 1 - \beta q$.

Proof. (1) : Note that, by (1.1), $\dim(\mathcal{Z}(f)) < \frac{2}{q}(1 - \beta q)$ if and only if there exists $\alpha < \frac{2}{q}(1 - \beta q)$ such that $C_\alpha(\mathcal{Z}(f)) = 0$. If $C_\alpha(\mathcal{Z}(f)) = 0$, by Lemma 2.2, there is no $S \in A_{-\beta}^q(\mathbb{T}) \setminus \{0\}$ such that $\text{supp}(S) \subset \mathcal{Z}(f)$. So, by Proposition 2.5 (1), f is cyclic in $A_\beta^p(\mathbb{T})$.

(2) : Suppose that $C_{1-\beta q}(\mathcal{Z}(f)) > 0$. There exists a probability measure μ of energy $I_{1-\beta q}(\mu) < \infty$, such that $\text{supp}(\mu) \subset \mathcal{Z}(f)$. So $\mu \in A_{-\beta q/2}^2(\mathbb{T}) \setminus \{0\}$. Since $|\hat{\mu}(n)| \leq 1$ for all $n \in \mathbb{Z}$ and $q \geq 2$, we have $\mu \in A_{-\beta}^q(\mathbb{T})$. By proposition 2.5 (2), f is not cyclic in $A_\beta^p(\mathbb{T})$.

(3) : Let $\frac{2}{q}(1 - \beta q) < \alpha \leq 1$. There exists $\varepsilon > 0$ such that $\frac{2}{q}(1 - \beta q) + \varepsilon < \alpha$. Let q' such that $\frac{2}{q} - 2\beta + \varepsilon = \frac{2}{q'}$. Since $\beta > \frac{1}{q} - \frac{1}{q'}$, by Lemma 2.1, $A^{q'}(\mathbb{T}) \subset A_{-\beta}^q(\mathbb{T})$. By Theorem 3.1, as q' satisfies $q' > \frac{2}{\alpha}$, there exists a closed subset $E \subset \mathbb{T}$ such that $\dim(E) = \alpha$ and a non zero positive measure $\mu \in A^{q'}(\mathbb{T}) \subset A_{-\beta}^q(\mathbb{T})$ such that $\text{supp}(\mu) \subset E$. Now (3) follows from proposition 2.5.(2).

(4) : Let $k = [q/2]$. Suppose first $\frac{2}{q}(1 - \beta q)k > 1 - 2(k + 1)\beta$ and let $0 < \varepsilon' < \varepsilon$ satisfying $1 - 2(k + 1)\beta \leq \frac{2}{q}(1 - \beta q)k - \varepsilon'$. Consider the set S_λ where λ verifies

$$\frac{2}{q}(1 - \beta q)k - \varepsilon' < \frac{1 - \lambda}{1 + \lambda} < \frac{2}{q}(1 - \beta q)k.$$

By Lemma 3.3.(3) we have $\dim(S_\lambda) = \frac{1-\lambda}{1+\lambda}$ and $C_{\frac{1-\lambda}{1+\lambda}}(k \times S_\lambda) = 0$. Therefore by Theorem 3.2.(a), every $f \in A_\beta^1(\mathbb{T})$ such that $\mathcal{Z}(f) = S_\lambda$ is cyclic in $A_\beta^p(\mathbb{T})$.

Now suppose $\frac{2}{q}(1 - \beta q)k \leq 1 - 2(k + 1)\beta$. We consider S_λ where $\frac{1-\lambda}{1+\lambda} = 1 - 2(k + 1)\beta$. By lemma 3.3.(3) we have $\dim(S_\lambda) = \frac{1-\lambda}{1+\lambda} = 1 - 2(k + 1)\beta$ and $C_{\frac{1-\lambda}{1+\lambda}}((k + 1) \times S_\lambda) = 0$. So by Theorem 3.2.(b), every $f \in A_\beta^1(\mathbb{T})$ such that $\mathcal{Z}(f) = S_\lambda$ is cyclic in $A_\beta^p(\mathbb{T})$.

Suppose now that $p = \frac{2k}{2k-1}$ for some $k \in \mathbb{N}^*$. As before, we consider S_λ where $\frac{1-\lambda}{1+\lambda} = 1 - 2k\beta = 1 - \beta q$. So again by Theorem 3.2.(b), every $f \in A_\beta^1(\mathbb{T})$ such that $\mathcal{Z}(f) = S_\lambda$ is cyclic in $A_\beta^p(\mathbb{T})$. \square

Note that the set E which is considered in 3.4.(4) verifies $C_\alpha(E) = 0$ where

$$\alpha \geq \max \left(\frac{2}{q}(1 - \beta q)k - \varepsilon, 1 - 2(k + 1)\beta \right).$$

4. PROOF OF THEOREM B

In this section we investigate the sharpness of the constant $\frac{2}{q}(1 - \beta q)$ in Theorem A.

Before proving Theorem B, we need the following two results. The following Lemma is an extension of Newman's Lemma 3 (see [10] pp 654-655).

Lemma 4.1. *Let $p \in [1, 2]$, $\beta \geq 0$ such that $\beta q \leq 1$. There exists $C > 0$ such that for all $f \in A_1^2(\mathbb{T})$,*

$$\|f\|_{A_\beta^p(\mathbb{T})} \leq C^{\frac{1}{p}} \|f\|_{A^2(\mathbb{T})}^{\frac{3}{2} - \frac{1}{p} - \beta} (\|f\|_{A^2(\mathbb{T})} + \|f'\|_{A^2(\mathbb{T})})^{\frac{1}{p} - \frac{1}{2} + \beta}.$$

Proof. It suffices to show that there exists $C > 0$ such that for all sequences $(c_n) \in \mathbb{C}^{\mathbb{N}^*}$,

$$\sum_{n=1}^{\infty} |c_n|^p (1 + |n|)^{p\beta} \leq C \left(\sum_{n=1}^{\infty} |c_n|^2 \right)^{\frac{3}{4}p - \frac{1}{2} - \frac{\beta p}{2}} \left(\sum_{n=1}^{\infty} n^2 |c_n|^2 \right)^{\frac{1}{2} - \frac{p}{4} + \frac{\beta p}{2}}.$$

Then we apply this inequality to $(\widehat{f}(n))_{n \geq 1}$ and $(\widehat{f}(-n))_{n \geq 1}$. Let $x^2 = \sum_{n \geq 1} |c_n|^2$ and $x^2 y^2 = \sum_{n \geq 1} n^2 |c_n|^2$. Note that $y \geq 1$. On one hand, by the Hölder inequality,

$$\begin{aligned} \sum_{1 \leq n \leq y} |c_n|^p (1 + n)^{p\beta} &\leq \left(\sum_{n=1}^y |c_n|^2 \right)^{p/2} \left(\sum_{n=1}^y (1 + n)^{\frac{2p\beta}{2-p}} \right)^{1-p/2} \\ &\leq (x^2)^{\frac{p}{2}} \left(y(1 + y)^{\frac{2p\beta}{2-p}} \right)^{1-\frac{p}{2}} \leq 2^{\beta p} x^p y^{1-\frac{p}{2}+p\beta}. \end{aligned}$$

On the other hand we set $\gamma = \frac{2p(1-\beta)}{2-p}$. Since $\beta q \leq 1$, $\gamma > 1$ and again by the Hölder inequality we obtain,

$$\begin{aligned} \sum_{n>y} |c_n|^p (1+n)^{p\beta} &\leq 2^p \left(\sum_{n>y} n^2 |c_n|^2 \right)^{\frac{p}{2}} \left(\sum_{n>y} (1+n)^{\frac{2p(\beta-1)}{2-p}} \right)^{1-\frac{p}{2}} \\ &\leq 2^p (x^2 y^2)^{\frac{p}{2}} \left(\frac{1}{\gamma-1} \right)^{1-p/2} (y^{1-\gamma})^{1-\frac{p}{2}} \\ &\leq 2^p \left(\frac{1}{\gamma-1} \right)^{1-p/2} x^p y^{1-\frac{p}{2}+p\beta} \end{aligned}$$

So the conclusion of the Lemma holds with

$$C = \max \left(2^{\beta p}, 2^p \left(\frac{1}{\gamma-1} \right)^{1-p/2} \right)$$

which is a positive constant depending only on p and β . \square

The following theorem is due to Körner (see [7, Theorem 1.2]).

Theorem 4.2. *Let $h : [0, \infty) \rightarrow [0, \infty)$ be an increasing continuous function with $h(0) = 0$ and let $\phi : [0, \infty) \rightarrow [0, \infty)$ be a decreasing function. Suppose that*

- (1) $\int_1^\infty \phi(x)^2 dx = \infty$;
- (2) there exist $K_1, K_2 > 1$ such that for all $1 \leq x \leq y \leq 2x$,

$$K_1 \phi(2x) \leq \phi(x) \leq K_2 \phi(y);$$

- (3) there exists $\gamma > 0$ such that

$$\lim_{x \rightarrow \infty} x^{1-\gamma} \phi(x) = \infty;$$

- (4) there exist $0 < K_2 < K_3 < 1$ such that for all $t > 0$,

$$K_2 h(2t) \leq h(t) \leq K_3 h(2t).$$

Then there exists a probability measure μ with support of Hausdorff h -measure zero such that

$$|\hat{\mu}(n)| \leq \phi \left(\frac{1}{h(|n|^{-1})} \right) \left(\ln \left(\frac{1}{h(|n|^{-1})} \right) \right)^{1/2}, \quad \forall n \neq 0.$$

Recall Theorem B reformulated in $A_\beta^p(\mathbb{T})$ space.

Theorem 4.3. *Let $1 < p < 2$ and $\beta \geq 0$ such that $\beta q < 1$.*

- (1) *If $f \in A_\beta^1(\mathbb{T})$ and $\mathcal{Z}(f)$ has strong α -measure 0 where $\alpha = \frac{2}{q}(1 - \beta q)$ then f is cyclic in $A_\beta^p(\mathbb{T})$.*

- (2) For every $\gamma > \frac{2}{q}$, there exists a closed subset $E \subset \mathbb{T}$ such that every $f \in A_\beta^1(\mathbb{T})$ satisfying $\mathcal{Z}(f) = E$ is not cyclic in $A_\beta^p(\mathbb{T})$ and such that $H_h(E) = 0$ where $h(t) = \frac{t^\alpha}{\ln(e/t)^\gamma}$ with $\alpha = \frac{2}{q}(1 - \beta q)$.

Note that in (2), H_h is closed to H_α .

Proof. (1) : The proof of this result holds by using arguments analogous to those of Newman for $\beta = 0$ (see [10, Theorem 1]). Denote by (a_k, b_k) the complementary intervals of $\mathcal{Z}(f)$ arranged in non-increasing order of lengths and set

$$r_n = 2\pi - \sum_{k=0}^n (b_k - a_k).$$

The set $\mathcal{Z}(f)$ has strong α -measure 0 where $\alpha = \frac{2}{q}(1 - \beta q)$ so

$$\lim_{n \rightarrow \infty} r_n n^{\frac{1}{\alpha}-1} = 0.$$

Let $\varepsilon > 0$ and $n \in \mathbb{N}$ such that $r_n < \varepsilon n^{1-\frac{1}{\alpha}}$ and $\varepsilon n^{-\frac{1}{\alpha}} < 1$. Let the function ψ be given by

$$\psi(x) = \max \left(1 - \frac{n^{\frac{1}{\alpha}}}{\varepsilon} \rho(x), 0 \right), \quad x \in \mathbb{T},$$

where

$$\rho(x) = \text{dist} \left(x, \mathbb{T} \setminus \bigcup_{k=1}^n]a_k, b_k[\right).$$

Then

$$\begin{aligned} \|\psi\|_{A^2(\mathbb{T})}^2 &= \int_{\mathbb{T} \setminus \bigcup_{k=1}^n]a_k, b_k[} \psi(t)^2 \, dt + \sum_{k=1}^n \int_{a_k}^{b_k} \psi(t)^2 \chi_{\{\rho(x) \leq \varepsilon n^{-\frac{1}{\alpha}}\}}(t) \, dt \\ &\leq r_n + \sum_{k=1}^n 2\varepsilon n^{-\frac{1}{\alpha}} \leq 3\varepsilon n^{1-\frac{1}{\alpha}}. \end{aligned}$$

Moreover

$$\begin{aligned} \|\psi'\|_{A^2(\mathbb{T})}^2 &= \int_{\mathbb{T}} \psi'(t)^2 \, dt = \sum_{k=1}^n \int_{a_k}^{b_k} \psi'(t)^2 \chi_{\{\rho(x) \leq \varepsilon n^{-\frac{1}{\alpha}}\}}(t) \, dt \\ &\leq \sum_{k=1}^n \left(\frac{n^{\frac{1}{\alpha}}}{\varepsilon} \right)^2 2\varepsilon n^{-\frac{1}{\alpha}} \leq 2 \frac{n^{1+\frac{1}{\alpha}}}{\varepsilon}. \end{aligned}$$

Since $\varepsilon n^{-\frac{1}{\alpha}} < \frac{n^{\frac{1}{\alpha}}}{\varepsilon}$ and $\alpha = \frac{2}{q}(1 - \beta q)$, by Lemma 4.1,

$$\|\psi\|_{A_\beta^p(\mathbb{T})} \leq C^{\frac{1}{p}} \left(3\varepsilon n^{1-\frac{1}{\alpha}} \right)^{\frac{3}{4}-\frac{1}{2p}-\frac{\beta}{2}} \left(5 \frac{n^{1+\frac{1}{\alpha}}}{\varepsilon} \right)^{\frac{1}{2p}-\frac{1}{4}+\frac{\beta}{2}} \leq C' \varepsilon^{1-\frac{1}{p}-\beta}$$

where C and C' depend only on β and p . Note that $1 - \psi$ is a Lipschitz function and $\mathcal{Z}(f) \subset \mathcal{Z}(1 - \psi)$. We conclude by Lemma 2.7.

(2) : Let $\alpha = \frac{2}{q}(1 - \beta q)$ and $\gamma > \frac{2}{q}$. By Theorem 4.2 with $\phi(t) = (t \ln(et))^{-1/2}$ for $t \geq 1$ and $h(t) = \frac{t^\alpha}{\ln(e/t)^\gamma}$ for $t \in [0, \infty)$, there exists a probability measure μ with support of Hausdorff h -measure zero such that

$$|\hat{\mu}(n)| \leq \phi\left(\frac{1}{h(|n|^{-1})}\right) \left(\ln\left(\frac{1}{h(|n|^{-1})}\right)\right)^{1/2} \leq (|n|^\alpha \ln(e|n|)^\gamma)^{-1/2},$$

for $n \neq 0$. So

$$\begin{aligned} \sum_{n \neq 0} |\hat{\mu}(n)|^q (1 + |n|)^{-\beta q} &\leq C \sum_{n \neq 0} |n|^{-\alpha q/2 - \beta q} \ln(e|n|)^{-\gamma q/2} \\ &\leq C \sum_{n \neq 0} \frac{1}{|n| \ln(e|n|)^{\gamma q/2}} < \infty \end{aligned}$$

with C a positive constant. Hence $\mu \in A_{-\beta}^q(\mathbb{T})$. We set $E = \text{supp}(\mu)$. By lemma 2.5 the result is proved. \square

5. REMARKS

We say that $(\omega_n) \in \mathbb{R}^{\mathbb{Z}}$ is a weight if $w_n \geq 1$ and $\omega_{n+k} \leq C\omega_n\omega_k$ for all $k, n \in \mathbb{Z}$ and C a positive constant. For ω a weight and $1 \leq p < \infty$ we set

$$A_\omega^p(\mathbb{T}) = \left\{ f \in C(\mathbb{T}), \quad \|f\|_{A_\omega^p(\mathbb{T})}^p = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^p \omega_n^p < \infty \right\}.$$

Note that $\|fS\|_{A_\omega^p(\mathbb{T})} \leq \|f\|_{A_\omega^1(\mathbb{T})} \|S\|_{A_\omega^p(\mathbb{T})}$ for $f \in A_\omega^1(\mathbb{T})$ and $S \in A_\omega^p(\mathbb{T})$. So we have the same result as (2.1) to characterize cyclicity in $A_\omega^p(\mathbb{T})$ by norm.

When $\omega_n = O((1 + |n|)^\varepsilon)$ for all $\varepsilon > 0$, for example $\omega_n = \ln(e + |n|)^\beta$ where $\beta \geq 0$, we can show the same result as Lemma 2.7. So by noting that for all $p \geq 1$ and $\delta > 0$,

$$A_\delta^p(\mathbb{T}) \subset A_\omega^p(\mathbb{T}) \subset A^p(\mathbb{T})$$

we obtain by Theorem A the following result:

Theorem 5.1. *Let $1 < p < 2$ and $\omega = (\omega_n)_{n \in \mathbb{Z}}$ a weight satisfying $\omega_n = O((1 + |n|)^\varepsilon)$ for all $\varepsilon > 0$.*

(1) *If $f \in A_\omega^1(\mathbb{T})$ and $\dim(\mathcal{Z}(f)) < \frac{2}{q}$ then f is cyclic in $A_\omega^p(\mathbb{T})$.*

- (2) For $\frac{2}{q} < \alpha \leq 1$, there exists a closed subset $E \subset \mathbb{T}$ such that $\dim(E) = \alpha$ and every $f \in A_\omega^1(\mathbb{T})$ satisfying $\mathcal{Z}(f) = E$ is not cyclic in $A_\omega^p(\mathbb{T})$.
- (3) For all $0 < \varepsilon < 1$, there exists a closed subset $E \subset \mathbb{T}$ such that $\dim(E) = 1 - \varepsilon$ and every $f \in A_\omega^1(\mathbb{T})$ satisfying $\mathcal{Z}(f) = E$ is cyclic in $A_\omega^p(\mathbb{T})$.

Proof. (1) : Let $f \in A_\omega^1(\mathbb{T})$ such that $\dim(\mathcal{Z}(f)) < \frac{2}{q}$. There exists $0 < \delta < 1/2$ such that $\dim(\mathcal{Z}(f)) < \frac{2}{q}(1 - \delta q)$. By Theorem 3.4.(1), every $g \in A_\delta^1(\mathbb{T})$ satisfying $\mathcal{Z}(g) = \mathcal{Z}(f)$ is cyclic in $A_\delta^p(\mathbb{T})$. Therefore by Lemma 2.7, there exist (f_n) a sequence of Lipschitz functions which are zero on $\mathcal{Z}(f)$ and such that

$$\lim_{n \rightarrow \infty} \|f_n - 1\|_{A_\delta^p(\mathbb{T})} = 0.$$

Moreover $\omega_n = O((1 + |n|)^\delta)$ so

$$\lim_{n \rightarrow \infty} \|f_n - 1\|_{A_\omega^p(\mathbb{T})} = 0.$$

Again by Lemma 2.7 in $A_\omega^p(\mathbb{T})$, we obtain that f is cyclic in $A_\omega^p(\mathbb{T})$.

(2) : By the theorem of Salem (see Theorem 3.1 and Theorem 1.3.(2)), there exists a closed set $E \subset \mathbb{T}$ such that $\dim(E) = \alpha$ and every $f \in A^1(\mathbb{T})$ satisfying $\mathcal{Z}(f) = E$ is not cyclic in $A^p(\mathbb{T})$. Let $f \in A_\omega^1(\mathbb{T})$ such that $\mathcal{Z}(f) = E$. Since $f \in A^1(\mathbb{T})$, f is not cyclic in $A^p(\mathbb{T})$. However $\|\cdot\|_{A^p(\mathbb{T})} \leq \|\cdot\|_{A_\omega^p(\mathbb{T})}$ therefore f is not cyclic in $A_\omega^p(\mathbb{T})$.

(3) : Let $0 < \varepsilon < 1$ and $\beta > 0$ such that $1 - 2([q/2] + 1)\beta \geq 1 - \varepsilon$. By Theorem 3.4.(4), there exists a closed set $E \subset \mathbb{T}$ such that

$$\dim(E) \geq 1 - 2([q/2] + 1)\beta \geq 1 - \varepsilon$$

and such that every $f \in A_\beta^1(\mathbb{T})$ satisfying $\mathcal{Z}(f) = E$ is cyclic in $A_\beta^p(\mathbb{T})$. Since $A_\beta^p(\mathbb{T}) \subset A_\omega^p(\mathbb{T})$, we obtain, by Lemma 2.7, that every $f \in A_\omega^1(\mathbb{T})$ satisfying $\mathcal{Z}(f) = E$ is cyclic in $A_\omega^p(\mathbb{T})$. \square

When $p > 2$ the search for cyclic vectors in $A^p(\mathbb{T})$ seems extremely difficult. Newman in [10] shows that for all $\alpha < 2\pi$ there exists $E \subset \mathbb{T}$ which has a Lebesgue measure $|E| > \alpha$ and such that every $f \in A^1(\mathbb{T})$ satisfying $\mathcal{Z}(f) = E$ is cyclic in $A^p(\mathbb{T})$. See also [10, Theorem 6] for the existence of non cyclic functions under some conditions. We also have a characterization of the cyclic vectors in term of the zeros of the Fourier transform when $p > 2$ but it's not very effective : A function $f \in A^1(\mathbb{T})$ is cyclic in $A^p(\mathbb{T})$ if and only if $\mathcal{Z}(f)$ does not support any non-zero function $g \in A^q(\mathbb{T})$ where $q = \frac{p}{p-1}$.

When $\omega_n = \log(e + |n|)^\beta$ where $0 < \beta < 1$, for all $p > \frac{2}{1-\beta}$ and for all $\alpha < 2\pi$, Nikolskii shows in [12, Corollary 6], there exists $E \subset \mathbb{T}$ which has a Lebesgue measure $|E| > \alpha$ and such that every $f \in A_\beta^1(\mathbb{T})$ satisfying $\mathcal{Z}(f) = E$ is cyclic in $A_\beta^p(\mathbb{T})$.

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